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The Problem of the Selection of an A-Posteriori Error Indicator Based on Smoothening Techniques

by

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Technical Note BN-1126

August 1991

INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Note BN-1126	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) The Problem of the Selection of an A-Posteriori Error Indicator Based on Smoothening Techniques		5. TYPE OF REPORT & PERIOD COVERED Final Life of the Contract
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Ivo M. Babuska[1] - Rodolfo Rodriguez[2]		8. CONTRACT OR GRANT NUMBER(s) [1]ONR N00014-90-J-1030 [2]NSF CCR-88-20279
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Physical Science and Technology University of Maryland College Park, MD 20742-2431		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, VA 22217		12. REPORT DATE August 1991
		13. NUMBER OF PAGES 35
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

THE PROBLEM OF THE SELECTION OF AN A-POSTERIORI ERROR INDICATOR BASED ON SMOOTHENING TECHNIQUES

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August 1991.

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Abstract: This paper adresses the problem of assesing the quality of an a-posteriori error estimate of a finite element solution. An error estimate based on local L^2-projections is analyzed in the case of translation invariant meshes. It is shown that for general meshes this technique does not lead to an asymptotically exact estimator. The problem is analyzed in detail in the one-dimensional setting. It is shown that an asymptotically exact estimator is not the optimal one when the solution is not sufficiently smooth. An optimal estimator for adaptively constructed meshes is given. Finally, a general mathematical framework for the quality assesment of estimators is introduced.

1. Introduction. Since the first papers by Babuška and Rheinboldt [5,6] on a-posteriori error estimates in the finite element method, the subject has become increasingly important in finite element practice [9,10,12,14,16,19,20,23,25,30,32]. Presently there exist various codes (for research or for commercial use) including a-posteriori error estimation.

There are essentially three major types of estimators (see also the discussion in [27]):

a) estimators based on residual considerations,

b) estimators based on averaging techniques,

c) estimators based on extrapolation.

The first proposed estimators were of residual type. For some of them a rather precise mathematical analysis has been presented [1,4,11,17,18,23]. Others were derived on purely heuristic grounds. Estimators of this type have been also used as the basis for adaptive approaches (see e.g. [8,16,22,23,25]). The first research adaptive code based on a residual estimator is likely FEARS [7,20].

In recent years, the estimators of type (b) have become extremely popular; in particular, the estimator proposed by Zienkiewicz and Zhu [31]. This paper will address estimators of this type in detail.

Estimators of type (c) are usually used in connection with the p-version of FEM (see e.g. [26,27]), while estimators of type (a) and (b) are used mostly but not exclusively in connection with the h-version of the method.

2. The model problems. We shall consider the two following model problems.

a) Let $\Omega \subset \mathbf{R}^2$ be a bounded polygonal domain with boundary Γ. We assume that $\Gamma = \Gamma_d \cup \Gamma_n$ (for simplicity we assume that Γ_d has positive length). We are interested in the solution of the problem

$$(2.1)\qquad \begin{cases} -\Delta u = f\,, & \text{in } \Omega\,, \\ u = 0\,, & \text{on } \Gamma_d\,, \\ \frac{\partial u}{\partial n} = g\,, & \text{on } \Gamma_n\,; \end{cases}$$

we denote by $\mathbf{n}$ the outer normal vector to Γ. We assume that f and g are sufficiently smooth functions; then the solution of (2.1) exists and is unique.

b) Let $\Omega_s := (0,1) \times (0,1)$; we consider the Laplace equation with periodic boundary conditions:

$$(2.2)\qquad \begin{cases} -\Delta u = f\,, \quad \text{in } \Omega_s\,, \\ u(0,x_2) = u(1,x_2)\,, \quad \frac{\partial u}{\partial x_1}(0,x_2) = \frac{\partial u}{\partial x_1}(1,x_2)\,, \quad 0 \le x_2 \le 1\,, \\ u(x_1,0) = u(x_1,1)\,, \quad \frac{\partial u}{\partial x_2}(x_1,0) = \frac{\partial u}{\partial x_2}(x_1,1)\,, \quad 0 \le x_1 \le 1\,, \\ \int_{\Omega_s} u \; dx_1 dx_2 = 0\,. \end{cases}$$

We assume that f is sufficiently smooth and that $\int_{\Omega_s} f \; dx_1 dx_2 = 0$; then the solution of (2.2) exists and is unique. If f and u are periodically extended to $\mathbf{R}^2$ then obviously u solves the differential equation in $\mathbf{R}^2$ also. From now on, when we speak about the regularity of the solution of this problem, we mean the regularity of the periodically extended solution.

Remark 2.1. We restrict ourselves to these model problems for the sake of simplicity but our results hold in more general situations. □

Let $|u|_1^2 := \int_\Omega |\nabla u|^2 dx_1 dx_2$ and $|u|_0^2 := \int_\Omega |u|^2 dx_1 dx_2$; let $H^1(\Omega) := \{u \ : \ |u|_1^2 + |u|_0^2 < \infty\}$ be the usual Sobolev space. Further, let us define $H_0^1(\Omega) := \{u \in H^1(\Omega) \ : \ u = 0 \ , \text{ on } \Gamma_d\}$ and $H^1_{\text{PER}}(\Omega_s) := \{u \in H^1(\Omega_s) \ : \ u \text{ has period 1 and } \int_{\Omega_s} u \ dx_1 dx_2 = 0\}$. Obviously $|\cdot|_1$ is a norm on $H_0^1(\Omega)$ and on $H^1_{\text{PER}}(\Omega_s)$.

We consider the usual variational form of (2.1) and (2.2). Let

$$B(u,v) := \int_\Omega \nabla u \cdot \nabla v \ dx_1 dx_2 \tag{2.3}$$

and

$$\|u\|_E^2 := B(u,u) = |u|_1^2 \ . \tag{2.4}$$

Further, for problem (2.1), let

$$F(v) := \int_\Omega fv \ dx_1 dx_2 \ + \ \int_{\Gamma_n} gv \ ds \tag{2.5}$$

and for problem (2.2) let

$$F(v) := \int_{\Omega_s} fv \ dx_1 dx_2 \ ; \tag{2.6}$$

both are linear functionals on $H_0^1(\Omega)$ and $H^1_{\text{PER}}(\Omega_s)$ respectively. The weak solution $u_0 \in H_0^1(\Omega)$ (respectively $u_0 \in H^1_{\text{PER}}(\Omega_s)$) is such that

$$B(u_0, v) = F(v) \ , \quad \forall v \in H_0^1(\Omega) \quad (\text{resp. } v \in H^1_{\text{PER}}(\Omega_s)) \ . \tag{2.7}$$

For the finite element method we partition Ω into a set of (closed) elements τ defined by the mesh $\mathcal{T}_h$; (to fix ideas, let us consider a triangular mesh). Let the mesh be regular in the sense of a minimal angle condition and let h denote the $\max_{\tau \in \mathcal{T}_h} \operatorname{diam}(\tau)$.

Furthermore let

$$S_h^r(\Omega) := \{u \in H^1(\Omega) \ : \ u|_\tau \in \mathcal{P}_r \ , \ \forall \tau \in \mathcal{T}_h\} \ ,$$

($\mathcal{P}_r$ denote the set of polynomials of degree r) and let

$$S_{h,0}^r(\Omega) := S_h^r(\Omega) \cap H_0^1(\Omega) \qquad \text{and} \qquad S_{h,\text{PER}}^r(\Omega_s) := S_h^r(\Omega) \cap H^1_{\text{PER}}(\Omega_s) \ .$$

The functions of $S^r_{h,\text{PER}}(\Omega_s)$ can be periodically extended to $\mathbf{R}^2$; we denote $S^r_{h,\text{PER}}$ to the set of these extensions.

We say that a mesh is (h_1, h_2)-*translation invariant* (h_1, h_2 positive numbers) if for any pair of integers ν_1, ν_2 and for any $u \in S^r_{h,\text{PER}}$, $u(x_1 + \nu_1 h_1, x_2 + \nu_2 h_2) \in S^r_{h,\text{PER}}$. Figure 2.1 shows some of these meshes.

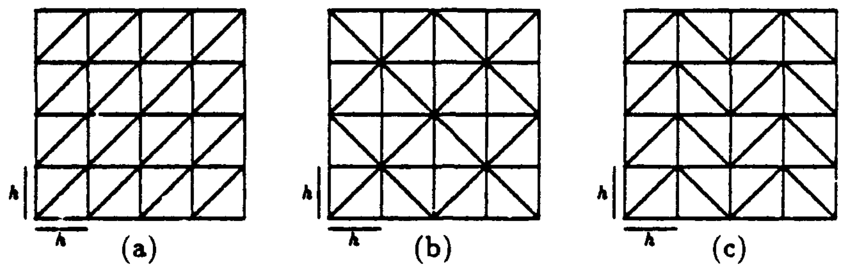

Figure 2.1

a) (h, h)-translation invariant mesh,

b) $(2h, 2h)$-translation invariant mesh,

c) $(2h, h)$-translation invariant mesh.

The finite element solution $u_{\text{FE}} \in S^r_{h,0}(\Omega)$ (resp. $S^r_{h,\text{PER}}(\Omega_s)$) is defined as:

$$(2.8) \qquad B(u_{\text{FE}}, v) = F(v)\ , \quad \forall v \in S^r_{h,0}(\Omega) \quad (\text{resp. } v \in S^r_{h,\text{PER}}(\Omega_s))\ .$$

3. A–posteriori error estimates based on averaging techniques. The main idea behind the averaging techniques is to observe that $\nabla u_{\text{FE}} \notin S^r_h(\Omega)^2$ and to construct $\mathbf{U} \in S^r_h(\Omega)^2$ using only u_{FE} with the hope that

$$(3.1) \qquad |\nabla u_0 - \mathbf{U}|_0 \ll |\nabla u_0 - \nabla u_{\text{FE}}|_0\ .$$

If (3.1) holds then

$$(3.2) \qquad |\mathbf{U} - \nabla u_{\text{FE}}|_0 \approx \|u_0 - u_{\text{FE}}\|_E\ .$$

and we can define

$$(3.3) \qquad \varepsilon := |\mathbf{U} - \nabla u_{\text{FE}}|_0$$

as the a–posteriori error estimator (with respect to the energy norm).

Given a class of functions $H \subset H^1_0(\Omega)$ (respectively $H \subset H^1_{\text{PER}}(\Omega_s)$) we call ε a *correct estimator* on H if there exist constants C_1 and C_2 ($0 < C_1 \le C_2 < \infty$) such that

$$(3.4) \qquad C_1 \le \frac{\varepsilon}{\|u_0 - u_{\text{FE}}\|_E} \le C_2\ ;$$

whenever the solution u_0 belongs to H; these constants must be independent of the mesh-size but could depend on the minimal angle of the mesh and on the class H.

On the other hand, the estimator ε is called *asymptotically exact* on H if

$$\xi := \frac{\varepsilon}{\|u_0 - u_{FE}\|_E} \to 1 \ , \quad \text{as } h \to 0 \ , \tag{3.5}$$

whenever the solution u_0 belongs to H.

There are many suggestions in the literature for the construction of U (and hence of ε), mostly without any theoretical justification. Let us divide these constructions into a few basic groups:

1) global construction of U,
2) local construction of U,
3) semilocal construction of U.

Further we may distinguish between two other groups:

a) general construction of U,
b) construction of U based on the available information of the problem (i.e. the differential operator) under consideration.

In what follows we restrict ourselves mostly to the case $r = 1$ (i.e. piecewise linear elements) for simplicity. In this case the solution u_{FE} and U itself are characterized by their nodal values. Each nodal value of U depends only on the values of u_{FE} in some neighborhood of this nodal point. Different kinds of neighborhoods distinguish the groups 1, 2 and 3 mentioned above.

Let A be a nodal point of a mesh $\mathcal{T}_h$ and $s \geq 1$ an integer. For $s = 1$ we define

$$\mathcal{N}^1(A) := \bigcup \{\tau \in \mathcal{T}_h \ : \ \tau \ni A\}$$

and for $s \geq 1$ we define recursively

$$\mathcal{N}^{s+1}(A) := \bigcup_{B \text{ node of } \mathcal{N}^s(A)} \mathcal{N}^1(B) \ .$$

When U(A) is defined only by the values of u_{FE} on $\mathcal{N}^1(A)$ we say that the construction is local. When the values on $\mathcal{N}^s(A)$ for $s > 1$ are used, we say that it is semilocal. The construction is global if the values of u_{FE} on the whole domain Ω are used. Figure 3.1 illustrates these notions.

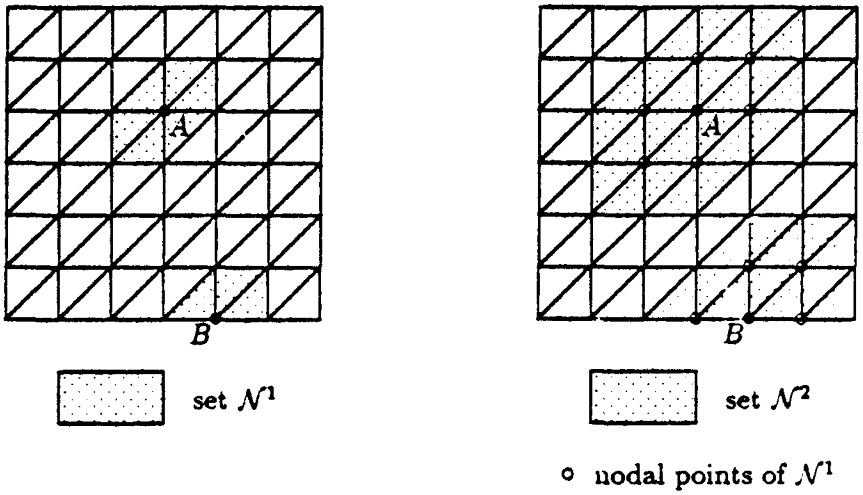

Figure 3.1

A typical construction of $\mathbf{U} := (U_1, U_2)$ consists of using the $L^2(\Omega)$-projection of $\frac{\partial u_{FE}}{\partial x_i}$ on $S_h^r(\Omega)$ as U_i, $i = 1, 2$ (see e.g. [31]). This construction belongs to the category 1-a; the matrix for determining the nodal values of U_i is the usual mass matrix. A local construction of type 2-a is obtained if the lumped mass matrix is used instead of the consistent one.

$U_i(A)$ can also be defined as the value at that node of the $L^2(\mathcal{N}^1(A))$-projection of $\frac{\partial u_{FE}}{\partial x_i}$ onto the set of continuous piecewise linear functions on that patch. This construction also belongs to category 2-a and will be especially addressed later.

In general any local construction defines $U_i(A)$ as a weighted average of the constant values of $\frac{\partial u_{FE}}{\partial x_i}$ in the elements $\tau \subset \mathcal{N}^1(A)$. For various types of averages we refer to [23,29].

These kind of constructions usually lead to estimators correct on $H_0^1(\Omega)$ (respectively on $H^1_{PER}(\Omega_s)$) in the sense of (3.4) with constants C_1, C_2 depending on the geometry of the mesh. However, asymptotic exactness is related with superconvergence effects. If the mesh is translation invariant then many superconvergence results hold; we refer to section 30 of [28] for this subject. These results can be easily employed for the construction of U and hence of the estimator.

Let us now consider problem (2.2). Assume that the mesh is (h_1, h_2)-translation invariant. Let

$$(D^1_{1,h}u)(x_1, x_2) := \frac{1}{h_1}\left[u(x_1 + \frac{h_1}{2}, x_2) - u(x_1 - \frac{h_1}{2}, x_2)\right] ,$$
$$(D^1_{2,h}u)(x_1, x_2) := \frac{1}{h_2}\left[u(x_1, x_2 + \frac{h_2}{2}) - u(x_1, x_2 - \frac{h_2}{2})\right] .$$

Then we have (see Theorem 30.1 of [28]):

THEOREM 3.1. *If the periodic extension of the solution u_0 of (2.2) has three bounded derivatives, i.e.:*

$$\left|\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k}(x_1,x_2)\right| \le M\ , \quad \forall (x_1,x_2)\in \mathbf{R}^2\ , \quad i,j,k=1,2\ , \tag{3.6}$$

and the mesh is (h_1,h_2)-translation invariant, then

$$\left|\left(\frac{\partial u_0}{\partial x_i} - D^1_{i,h}u_{\text{FE}}\right)(x_1,x_2)\right| \le C|\log h|h^2 M\ , \quad i=1,2\ .$$

Remark 3.1. Theorem 3.1 can be easily understood. $\frac{\partial u_0}{\partial x_i}$ is the solution of problem (2.2) with f replaced by $\frac{\partial f}{\partial x_i}$ and $D^1_{i,h}u_{\text{FE}}$ is the finite element solution of problem (2.2) with f replaced by $D^1_{i,h}f$. Since $\left|\frac{\partial f}{\partial x_i} - D^1_{i,h}f\right| = \mathcal{O}(h^2)$, then the conclusion of the theorem holds. (The $\log(h)$-term comes from the L^∞ error estimate of the finite element solution). □

This theorem allows us to design proper averages, for [illegible]s such as those in Fig. 2.1. For the following examples let us assume that the solution u_0 of problem (2.2) is in the subspace H of periodic functions with three bounded derivatives.

Mesh 2.1.a. Consider the set $\mathcal{N}^1(A)$ shown in Fig 3.2.

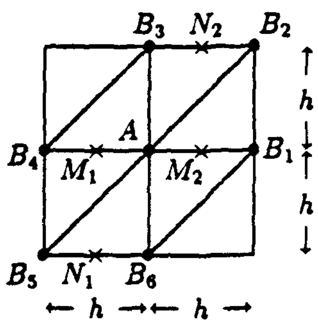

Figure 3.2

We have:

$$\frac{\partial u_0}{\partial x_1}(M_1) = \frac{\partial u_{\text{FE}}}{\partial x_1}(M_1) + |\log h|\mathcal{O}(h^2)\ ,$$
$$\frac{\partial u_0}{\partial x_1}(M_2) = \frac{\partial u_{\text{FE}}}{\partial x_1}(M_2) + |\log h|\mathcal{O}(h^2)\ ,$$
$$\frac{\partial u_0}{\partial x_1}(N_1) = \frac{\partial u_{\text{FE}}}{\partial x_1}(N_1) + |\log h|\mathcal{O}(h^2)\ ,$$
$$\frac{\partial u_0}{\partial x_1}(N_2) = \frac{\partial u_{\text{FE}}}{\partial x_1}(N_2) + |\log h|\mathcal{O}(h^2)\ .$$

But

$$\frac{\partial u_0}{\partial x_1}(A) = \frac{1}{2}\left[\frac{\partial u_0}{\partial x_1}(M_1) + \frac{\partial u_0}{\partial x_1}(M_2)\right] + \mathcal{O}(h^2)$$

and

$$\frac{\partial u_0}{\partial x_1}(A) = \frac{1}{2}\left[\frac{\partial u_0}{\partial x_1}(N_1) + \frac{\partial u_0}{\partial x_1}(N_2)\right] + \mathcal{O}(h^2)\ ,$$

and hence

$$\begin{aligned}\frac{\partial u_0}{\partial x_1}(A) = \frac{1}{6}\Bigg[& 2\frac{\partial u_{FE}}{\partial x_1}(M_1) + 2\frac{\partial u_{FE}}{\partial x_1}(M_2) \\ & + \frac{\partial u_{FE}}{\partial x_1}(N_1) + \frac{\partial u_{FE}}{\partial x_1}(N_2)\Bigg] + |\log h|\mathcal{O}(h^2)\ . \end{aligned} \tag{3.7}$$

The average of $\frac{\partial u_{FE}}{\partial x_1}$ on the right hand side of (3.7) is identical to the nodal value $U_1(A)$ obtained by the $L^2(\mathcal{N}^1(A))$-projection (or equivalently by the lumped $L^2(\Omega)$-projection). An analogous result is true for U_2. Therefore, since U is defined by its nodal values which are higher order approximations of the exact solution, an explicit calculation shows that (3.1) holds and hence, on these kind of meshes, this projection leads to an asymptotically exact estimator on the space H of periodic functions satisfying (3.6).

Mesh 2.1.b. In this case the mesh is $(2h, 2h)$–translation invariant. Consider $\mathcal{N}^1(A)$ and $\mathcal{N}^1(C)$ as shown in Fig 3.3.

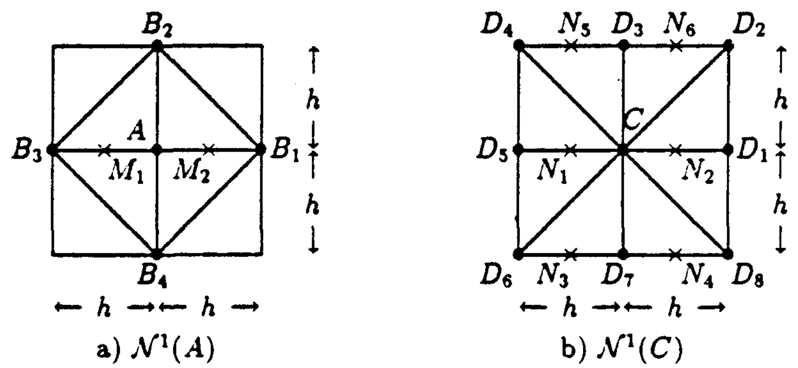

Figure 3.3

We have

$$\begin{aligned}\frac{\partial u_0}{\partial x_1}(A) &= \frac{1}{2h}\left[u_{FE}(B_1) - u_{FE}(B_3)\right] + |\log h|\mathcal{O}(h^2) \\ &= \frac{1}{2h}\left\{[u_{FE}(B_1) - u_{FE}(A)] + [u_{FE}(A) - u_{FE}(B_3)]\right\} + |\log h|\mathcal{O}(h^2) \\ &= \frac{1}{2}\left[\frac{\partial u_{FE}}{\partial x_1}(M_2) + \frac{\partial u_{FE}}{\partial x_1}(M_1)\right] + |\log h|\mathcal{O}(h^2)\ . \end{aligned} \tag{3.8}$$

Similarly for $\mathcal{N}^1(C)$:

$$
\begin{aligned}
\frac{\partial u_0}{\partial x_1}(A) = \frac{1}{8}\Bigg[& 2\frac{\partial u_{FE}}{\partial x_1}(N_1) + 2\frac{\partial u_{FE}}{\partial x_1}(N_2) + \frac{\partial u_{FE}}{\partial x_1}(N_3) + \frac{\partial u_{FE}}{\partial x_1}(N_4) \\
& + \frac{\partial u_{FE}}{\partial x_1}(N_5) + \frac{\partial u_{FE}}{\partial x_1}(N_6)\Bigg] + |\log h|\mathcal{O}(h^2)\,. \qquad (3.9)
\end{aligned}
$$

It is easy to check that the $L^2(\mathcal{N}^1)$-projection of $\frac{\partial u_{FE}}{\partial x_1}$ gives the same averages as in (3.8) and (3.9). Therefore, also for these kind of meshes, this projection leads to asymptotically exact estimators on the space H of periodic function with three bounded derivatives.

Mesh 2.1.c. Now the mesh is $(2h, h)$–translation invariant. Let $\mathcal{N}^1(A)$ be as in Fig 3.4.

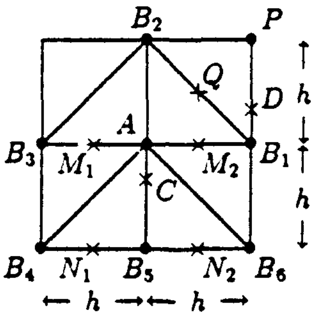

Figure 3.4

In this case, the value obtained by the $L^2(\mathcal{N}^1)$-projection is

$$
U_1(A) = \frac{1}{6}\left[2\frac{\partial u_{FE}}{\partial x_1}(M_1) + 2\frac{\partial u_{FE}}{\partial x_1}(M_2) + \frac{\partial u_{FE}}{\partial x_1}(N_1) + \frac{\partial u_{FE}}{\partial x_1}(N_2)\right]\,. \qquad (3.10)
$$

This value coincides up to higher order terms with $\frac{\partial u_0}{\partial x_1}(C)$ (see Fig. 3.3; dist$(AC) = \frac{h}{3}$) instead of $\frac{\partial u_0}{\partial x_1}(A)$. An analogous statement is true for each interior node; for instance, $U_1(B_1)$ coincides with $\frac{\partial u_0}{\partial x_1}(D)$ (see again Fig. 3.3; dist$(B_1 D) = \frac{h}{3}$) instead of $\frac{\partial u_0}{\partial x_1}(B_1)$.

Let R be the square of vertices A, B_1, P, B_2; by using the corresponding averages of the form (3.10) for each vertex and applying Theorem 3.1 and appropiate mean value theorems we obtain:

$$
\int_R \left|U_1 - \frac{\partial u_0}{\partial x_1}\right|^2 dx_1 dx_2 = \frac{h^4}{27}\left[\frac{\partial^2 u_0}{\partial x_1 \partial x_2}(Q)\right]^2 + |\log h|\mathcal{O}(h^5)\,.
$$

Therefore, U_1 is not an approximation of $\frac{\partial u_0}{\partial x_1}$ of higher order than $\frac{\partial u_{FE}}{\partial x_1}$. A similar analysis is valid for U_2. Hence the condition (3.1) on which the effectiveness of this estimator is based is not satisfied.

So, the estimator ε will not be asymptotically exact on these kind of meshes unless a fortunate compensation happens. But, in general, this is not the case. For instance consider the following problem:

$$\begin{cases} -\Delta u = 0 \ , & \text{in } \Omega \ , \\ u = g \ , & \text{on } \Gamma \ , \end{cases}$$

where Ω is the unit square and g is defined in Figure 3.5:

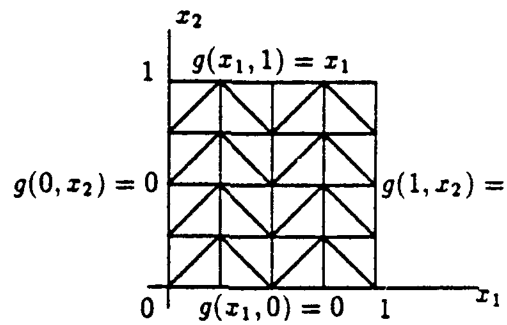

Figure 3.5

The solution of this problem is $u_0(x_1, x_2) = x_1 x_2$. In this example the F.E. approximation coincides with the exact solution at the nodes of the mesh (i.e. u_{FE} is the interpolant of u_0). So the exact error and the estimator can be explicitly computed. For any triangle τ of the mesh we have

$$\int_\tau |\nabla u_0 - \nabla u_{FE}|^2 = \frac{h^4}{6} \qquad \text{and} \qquad \int_\tau |\mathrm{U} - \nabla u_{FE}|^2 = \frac{7h^4}{54} \ .$$

Therefore

$$\xi := \frac{\varepsilon}{\|u_0 - u_{FE}\|_E} = \frac{\sqrt{7}}{3} \approx 0.88 \ ,$$

independently of the mesh size h. That is, in spite of the smoothness of the solution, $\xi \not\to 1$ as $h \to 0$.

This example shows that the estimator based on the $L^2(\mathcal{N}^1)$-projection is not always asymptotically exact, even in the simple case of translation invariant meshes. To obtain an asymptotically exact estimator, the geometry of the mesh should be used in a more sophisticated way.

Now, let us briefly consider the case $r = 2$. For (h_1, h_2)-invariant translation meshes, let

$$D^2_{i,h} u := \frac{4}{3} D^1_{i,h} u - \frac{1}{3} D^1_{i,2h} u \ .$$

For smooth functions u this is a fourth order approximation of $\frac{\partial u}{\partial x_i}$. The following theorem is also a consequence of Theorem 30.1 of [28].

THEOREM 3.2. *If the periodic extension of the solution u_0 of (2.2) has four bounded derivatives, i.e.:*

$$\left|\frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l}(x_1, x_2)\right| \leq M \ , \quad \forall (x_1, x_2) \in \mathbf{R}^2 \ , \quad i, j, k, l = 1, 2 \ ,$$

and the mesh is (h_1, h_2)-translation invariant, then

$$\left|\left(\frac{\partial u_0}{\partial x_i} - D^2_{i,h} u_{FE}\right)(x_1, x_2)\right| \leq C h^3 M \ , \quad i = 1, 2 \ .$$

By using this theorem we may construct $U \in S^2_h(\Omega)^2$ satisfying (3.1) for invariant translation meshes. For instance consider a uniform mesh as that in 2.1.a.

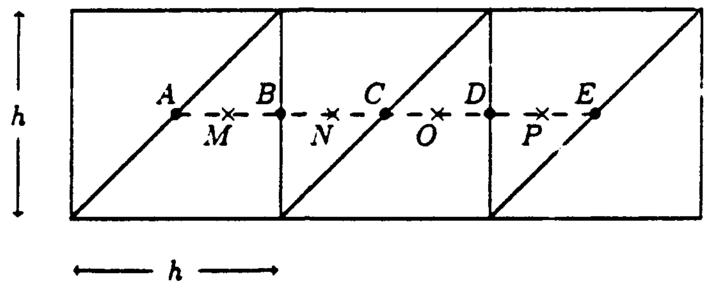

Figure 3.6

We need to define the nodal values of U. For instance, with the notation of Fig. 3.6, let

$$\begin{aligned} U_1(C) &:= (D^2_{1,h} u_{FE})(C) \\ &= \frac{4}{3}\frac{u_{FE}(D) - u_{FE}(B)}{h} - \frac{1}{3}\frac{u_{FE}(E) - u_{FE}(A)}{2h} \\ &= \frac{2}{3}\left[\frac{u_{FE}(D) - u_{FE}(C)}{h/2} + \frac{u_{FE}(C) - u_{FE}(B)}{h/2}\right] \\ &\quad - \frac{1}{12}\left[\frac{u_{FE}(E) - u_{FE}(D)}{h/2} + \frac{u_{FE}(D) - u_{FE}(C)}{h/2} + \frac{u_{FE}(C) - u_{FE}(B)}{h/2} + \frac{u_{FE}(B) - u_{FE}(A)}{h/2}\right] \\ &= \frac{7}{12}\left[\frac{\partial u_{FE}}{\partial x_1}(O) + \frac{\partial u_{FE}}{\partial x_1}(N)\right] - \frac{1}{12}\left[\frac{\partial u_{FE}}{\partial x_1}(P) + \frac{\partial u_{FE}}{\partial x_1}(M)\right] \ ; \end{aligned}$$

by using Theorem 3.2 we have

$$U_1(C) = \frac{\partial u_0}{\partial x}(C) + O(h^3).$$

The same construction can be made for any other node P of the mesh and also for U_2; i.e., in general,

$$U_i(P) := (D^2_{i,h} u_{FE})(P) \ .$$

It can be easily proved that this construction yields an asymptotically exact estimator.

We observe that in this case the construction of U is semilocal; in fact $U(C)$ is an average of $\frac{\partial u_{FE}}{\partial x_i}$, $i = 1, 2$ at points of triangles $\tau \subset \mathcal{N}^2(C)$.

In the case of problem (2.1) we have to extend the solutions u_0 and u_{FE} outside of Ω so that the averages can be computed also on the boundary. We suggest the use of Babič extension techniques. In the next section we shall discuss in detail these techniques for the one-dimensional case. For the n-dimensional case see, for instance, [21] (pp 75–77).

For general meshes it is probably impossible that estimators of this kind can be asymptotically exact, since this seems to depend on a superconvergence effect. At any rate, for practical constructions, those estimators that are asymptotically exact for invariant translation meshes seems to be preferable.

We addressed only local and semilocal constructions. However, it is very likely that global estimators based on L^2-projections will not bring a larger benefit and will not avoid the problem with general meshes.

So far we have discussed only estimators of type (a). Semilocal or global constructions of U of type (b) are more expensive but could lead to asymptotically exact estimators also for general meshes. These constructions are based on postprocessing techniques, but we will not analyze these kind of estimators here. For further references on this subject see [15,23].

Remark 3.2. It is very likely that the first result related to the superconvergence of smoothening constructions that could be used to define a-posteriori estimators is in [24] (1969). An important result of the same type is also in [13]. □

4. Model problem in one dimension. In the forthcoming sections we shall analyze in detail the one dimensional case to gain additional insight. Let us consider the problem

$$\begin{cases} -u'' = f\ , & \text{on } I := (0,1)\ , \\ u(0) = u(1) = 0 \end{cases} \tag{4.1}$$

and let f be such that $u \in H_0^1(I)$.

Let $\mathcal{T}$ be a partition of the interval I with nodes $0 = x_0 < x_1 < \cdots < x_n = 1$; $I_i := [x_{i-1}, x_i]$, $h_i := x_i - x_{i-1}$, $i = 1, \ldots, n$.

Consider the finite element method with piecewise linear elements on the mesh $\mathcal{T}$. In this case we have

$$u_{FE}(x_i) = u(x_i)\ , \quad i = 1, \ldots, n\ ;$$

i.e. u_{FE} is the piecewise linear interpolant of u.

Denote $e_u := u - u_{FE}$. We shall be interested in error indicators η_i approximating $|e_u|_{1,I_i} := |e'_u|_{0,I_i} := \left[\int_{I_i} (e'_u)^2 dx\right]^{\frac{1}{2}}$, where

$$\eta_i^2 := \int_{I_i} (U - u'_{FE})^2\ ; \tag{4.2}$$

here U is a continuous piecewise linear function on I. In Fig. 4.1 we show u', u'_{FE} and $U|_{I_i}$.

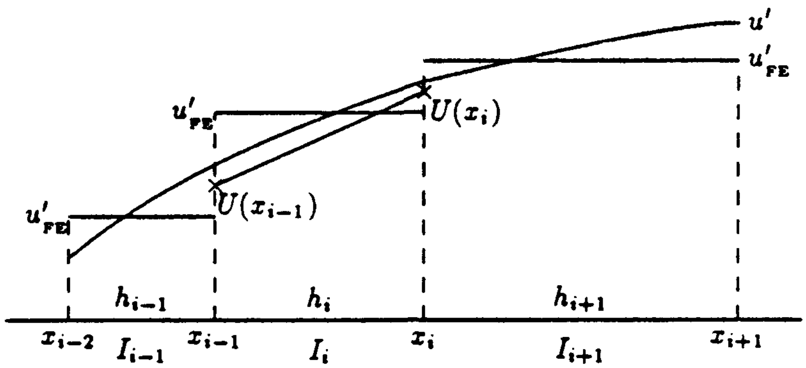

Figure 4.1

We shall consider only local constructions of U and discuss various aspects of such constructions. For any x_i, $1 \leq i \leq n-1$, we define

(4.3)
$$U(x_i) := \sum_{j=i-1}^{j=i+1} \gamma_j^{(i)} u_{FE}(x_j) \ ,$$

where $\gamma_j^{(i)}$ depend on the mesh.

We constrain the coefficients $\gamma_j^{(i)}$ so that $U(x_i) = u'_{FE}(x_i)$ when u'_{FE} is linear on $\mathcal{N}(x_i) := I_i \cup I_{i+1}$. This leads to the formula

(4.4)
$$\begin{aligned} U(x_i) &= \frac{u_{FE}(x_i) - u_{FE}(x_{i-1})}{h_i} + \alpha_i \left[\frac{u_{FE}(x_{i+1}) - u_{FE}(x_i)}{h_{i+1}} - \frac{u_{FE}(x_i) - u_{FE}(x_{i-1})}{h_i} \right] \\ &= u'_{FE}|_{I_i} + \alpha_i \left[u'_{FE}\right]_i = u'_{FE}|_{I_{i+1}} + \beta_i \left[u'_{FE}\right]_i \ , \end{aligned}$$

where

(4.5)
$$\left[u'_{FE}\right]_i := u'_{FE}|_{I_{i+1}} - u'_{FE}|_{I_i}$$

is the jump of the derivative of u_{FE} at x_i; (obviously $\alpha_i = 1 + \beta_i$). $U(x_i)$ depends only on the values of u'_{FE} in the elements I_i and I_{i+1} which contains the mesh point x_i.

For the values of $U(x_0)$ and $U(x_n)$ formula (4.4) cannot be directly applied. To use this formula we need to extend u and u_{FE} for $x < 0$ (and for $x > 1$) by a process that preserves the smoothness of the function under consideration. For example, it is possible to use Babič extension (see e.g. [21], pp 75–77). This extension preserves the smoothness of u in the sense that if $u \in H^s([0,x_2])$, $(1 \leq s \leq 3)$, then the extended function v on $[x_{-1},x_2]$ which coincides with u on $[x_0,x_2]$ is such that $v \in H^s([x_{-1},x_2])$ and $\|v\|_{H^s([x_{-1},x_2])} \leq C\|u\|_{H^s([x_0,x_2])}$, where C depends only on the local regularity of the mesh. Using $x_1 = -h_2$, $v(x_{-1})$ depends only on $u(x_0)$, $u(x_1)$ and $u(x_2)$ and hence the extension can be used for u_{FE} as well. When $1 \leq s \leq 2$ the simple antisymmetric extension (that is a Babič extension preserving at most H^2-smoothness) can be used. In any case, (4.4) can be applied to the extended function. Let us remark that if u is extended antisymmetrically $U(x_0) = u'_{FE}(x_0)$.

Now, defining

$$V(x) := U(x) - u'_{FE}(x)\ , \quad x \in I\ , \tag{4.6}$$

we have for $x \in I_i$

$$V(x) = V_i(x) := \beta_{i-1}\left[u'_{FE}\right]_{i-1}\varphi_{i-1}(x) + \alpha_i\left[u'_{FE}\right]_i\varphi_i(x)\ , \tag{4.7}$$

where φ_i and φ_{i-1} are the standard linear shape functions on I_i. Hence using (4.2) we get

$$\eta_i^2 = |V_i|^2_{0,I_i} = \int_{I_i}[V_i(x)]^2dx \tag{4.8}$$

and so we can understand the indicator η_i^2 as a quadratic form on the jumps $\left[u'_{FE}\right]_{i-1}$ and $\left[u'_{FE}\right]_i$; i.e.:

$$\eta_i^2 = h_i\sum_{j,k=i-1,i}\gamma_{j,k}\left[u'_{FE}\right]_j\left[u'_{FE}\right]_k\ . \tag{4.9}$$

Using (4.7) and (4.8) $\gamma_{j,k}$ are directly related to β_{i-1} and α_i in an obvious way. However the indicator (4.9) can be defined without any relation to the function $V_i(x)$. We can simply define $\gamma_{j,k}$ and address the accuracy of that indicator.

We define the error estimator:

$$\varepsilon^2 := \sum_{i=1}^{n}\eta_i^2\ . \tag{4.10}$$

Let us mention that the well known Zienkiewicz-Zhu estimator (Z-Z) [31] which uses the L^2-projection of u'_{FE} on $\mathcal{N}(x_i)$ leads to

$$\beta_{i-1} = -\frac{h_{i-1}}{h_{i-1}+h_i}\ , \qquad \alpha_i = \frac{h_{i+1}}{h_i+h_{i+1}}\ . \tag{4.11}$$

On the other hand, the Babuška-Miller estimator [4] (B-M) leads to

$$\beta_{i-1} = -\frac{h_i}{h_{i-1}+h_i}\ , \qquad \alpha_i = \frac{h_i}{h_i+h_{i+1}}\ . \tag{4.12}$$

Obviously for uniform meshes both estimators, Z-Z and B-M, are identical.

By using ficticious nodes $x_1 := -h_2$ and $x_{n+1} := 1 + h_{n-1}$, the Babič extension of u preserving H^s-smoothness for $1 \leq s \leq 3$, gives $\left[u'_{FE}\right]_0 = \left[u'_{FE}\right]_1$ and $\left[u'_{FE}\right]_n = \left[u'_{FE}\right]_{n-1}$. The Z-Z estimator uses $\left[u'_{FE}\right]_0 = \left[u'_{FE}\right]_n = 0$ (these are also the values obtained by the antisymmetric Babič extension). Nevertheless it is easy to see that both extensions lead to the same indicator.

5. Principles of the assesment of the quality of an error estimator. The formulation of the quality of an error estimator is far from being obvious. One of the most natural principles is to compare the indicator η_i and the true error $|e_u|_{1,I_i}$. To this end we define the *elemental effectivity indices*

$$\xi_i := \frac{\eta_i}{|e_u|_{1,I_i}} , \qquad i = 1, \ldots, n . \tag{5.1}$$

We say that the indicator is *asymptotically exact* if for any mesh such that $0 < C_1 \leq \frac{h_i}{h_{i+1}} \leq C_2 < \infty$, $(i = 1, \ldots, n-1)$, and, for u sufficiently smooth and such that $|e_u|_{1,I_i} \geq Ch_i^{\frac{3}{2}}$, $(C > 0)$, we have for all $i = 1, \ldots, n$,

$$\xi_i \to 1 , \qquad \text{as} \qquad h_i \to 0 .$$

If these elemental effectivity indices are used as a criterion to choose an indicator, the asymptotically exact indicators would be preferable when the mesh is sufficiently refined. However we note that this obvious criterion needs to be refined as it will be seen later.

In practice, the solution u is unknown but, usually, it is known that u belongs to certain space of functions $\mathcal{H} \subset H_0^1(I)$. In this case we may define

$$(\mathcal{L}u)(x) := (\mathcal{L}_i u)(x) := (e_u' - V_i)(x) , \qquad x \in I_i , \tag{5.2}$$

and understand $\mathcal{L}_i$ as a mapping of $\mathcal{H}$ onto the space $L^2(I_i)$. Since

$$\Big| |e_u|_{1,I_i} - \eta_i \Big| = \Big| |e_u'|_{0,I_i} - |V_i|_{0,I_i} \Big| \leq |e_u' - V_i|_{0,I_i} = |\mathcal{L}_i u|_{0,I_i} , \tag{5.3}$$

then $|\mathcal{L}_i u|_{0,I_i}$ can be used for the assesment of the quality of the indicator.

We could introduce

$$\rho(\mathcal{H}) := \sup_{v \in \mathcal{H}} \frac{|\mathcal{L}_i v|_{0,I_i}}{|e_v|_{1,I_i}}$$

as a measure of this quality. For any indicator, there exists a function $\hat{v} \in \mathcal{H}$ such that

$$|\mathcal{L}_i \hat{v}|_{0,I_i} = \rho(\mathcal{H}) |e_{\hat{v}}|_{1,I_i} .$$

If we do not know anything more about the solution than $u \in \mathcal{H}$, the safest choice would be to use that indicator which minimizes $\rho(\mathcal{H})$. However this approach has a major disadvantage: $\rho(\mathcal{H}) < \infty$ only for very restrictive spaces $\mathcal{H}$. For example, even for $\mathcal{H} = C^\infty(I)$, $\rho(\mathcal{H}) = \infty$

To avoid this, we can asses the quality of an indicator by the norm of $\mathcal{L}_i$

$$\|\mathcal{L}_i\| := \sup_{v \in \mathcal{H}} \frac{|\mathcal{L}_i v|_{0,I_i}}{\|v\|_{\mathcal{H}}} , \tag{5.4}$$

and prefer those indicators leading to smaller $\|\mathcal{L}_i\|$. This quality measure obviously depends on the space $\mathcal{H}(I)$ of possible solutions. In subsequent sections we shall study this criterion in detail.

So far, we described criterions of elemental type. We can also define the quality of the indicator through the corresponding estimator, i.e. to use the global effectivity index

$$\xi := \frac{\varepsilon}{|e_u|_{1,I}} = \frac{\left(\sum_{i=1}^n \eta_i^2\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^n |e_u|_{1,I_i}^2\right)^{\frac{1}{2}}} \tag{5.5}$$

or analogously the norm of $\mathcal{L}$

$$\|\mathcal{L}\| := \sup_{u\in\mathcal{H}} \frac{|\mathcal{L}u|_{0,I}}{\|u\|_{\mathcal{H}}} = \frac{\left[\int_I (e_u' - V)^2\right]^{\frac{1}{2}}}{\|u\|_{\mathcal{H}}} \tag{5.6}$$

and to prefer an indicator based on the (global) quality assesment of the estimator.

6. Comparison of Z-Z and B-M indicators based on the elemental effectivity index. In this section we compare Z-Z and B-M indicators in order to show the difficulties arising in the comparison of any estimators.

THEOREM 6.1. *B-M indicator is asymptotically exact.*

Proof. It can be easily seen that when u is a quadratic polynomial, U coincides exactly with u' and hence $\xi_i = 1$. In general, for u sufficiently smooth (say, for instance, that $u'''(x)$ is bounded for $x \in \widetilde{I}_i$, where $\widetilde{I}_i := I_{i-1} \cup I_i \cup I_{i+1}$), let

$$u(x) = c_0 + c_1(x - x_i) + c_2(x - x_i)^2 + R(x)$$

be the Taylor expansion of u and let

$$\widetilde{u}(x) := c_0 + c_1(x - x_i) + c_2(x - x_i)^2 .$$

Let $\widetilde{u}_{FE}$ be the finite element aproximation of $\widetilde{u}$ (i.e. its interpolant). Let U and $\widetilde{U}$ be the linear functions on I_i defined by (4.4) by using B-M coefficients (4.12) for u_{FE} and $\widetilde{u}_{FE}$ respectively; i.e., for $j = i-1, i$,

$$\begin{aligned} U(x_j) &= \frac{h_j}{h_j + h_{j+1}} \frac{u(x_{j+1}) - u(x_j)}{h_{j+1}} + \frac{h_{j+1}}{h_j + h_{j+1}} \frac{u(x_j) - u(x_{j-1})}{h_j} ; \\ \widetilde{U}(x_j) &= \frac{h_j}{h_j + h_{j+1}} \frac{\widetilde{u}(x_{j+1}) - \widetilde{u}(x_j)}{h_{j+1}} + \frac{h_{j+1}}{h_j + h_{j+1}} \frac{\widetilde{u}(x_j) - \widetilde{u}(x_{j-1})}{h_j} . \end{aligned} \tag{6.1}$$

Since $\widetilde{u}$ is a quadratic polynomial, then $\widetilde{U}(x) = \widetilde{u}'(x)$, for $x \in I$ and hence

$$\left|\eta_i - |e_*|_{1,I_i}\right| = \left| |U - u'_{FE}|_{0,I_i} - |u' - u'_{FE}|_{0,I_i} \right| \le |U - u'|_{0,I_i} \le |U - \widetilde{U}|_{0,I_i} + |\widetilde{u}' - u'|_{0,I_i} \; ;$$

therefore,

$$|\xi_i - 1| = \left| \frac{\eta_i}{|e_*|_{1,I_i}} - 1 \right| \le \frac{|U - \widetilde{U}|_{0,I_i} + |\widetilde{u}' - u'|_{0,I_i}}{|e_*|_{1,I_i}} \; . \tag{6.2}$$

Now, since $u - \widetilde{u} = R$, then by using (6.1) we have for $j = i-1, i$,

$$\begin{aligned}(U - \widetilde{U})(x_j) &= \frac{h_j}{h_j + h_{j+1}} \frac{R(x_{j+1}) - R(x_j)}{h_{j+1}} + \frac{h_{j+1}}{h_j + h_{j+1}} \frac{R(x_j) - R(x_{j-1})}{h_j} \\ &= \frac{h_j}{h_j + h_{j+1}} R'(\zeta_{j+1}) + \frac{h_{j+1}}{h_j + h_{j+1}} R'(\zeta_j)\end{aligned}$$

with $x_{j-1} < \zeta_j < x_j < \zeta_{j+1} < x_{j+1}$; hence

$$|U - \widetilde{U}|_{0,I_i} \le h_i^{\frac{1}{2}} \max_{\xi \in \widetilde{I}_i} |R'(x)| \; .$$

On the other hand,

$$|u' - \widetilde{u}'|_{0,I_i} \le h_i^{\frac{1}{2}} \max_{\xi \in I_i} |R'(x)|$$

and therefore, using these bounds in (6.2), we obtain

$$|\xi_i - 1| \le \frac{2h_i^{\frac{1}{2}} \max_{\xi \in \widetilde{I}_i} |R'(x)|}{|e_*|_{1,I_i}} \; .$$

Because of the assumed smoothness of u and the regularity of the mesh (i.e. $0 < C_1 \le \frac{h_i}{h_{j+1}} \le C_2 < \infty$), $R'(x) = \mathcal{O}(h_i^2)$, for $x \in \widetilde{I}_i$; therefore, whenever $|e_*|_{1,I_i} \ge Ch_i^{\frac{3}{2}}$, $(C > 0)$, then $|\xi_i - 1| \to 0$ as $h_i \to 0$; that is, B-M indicator is asymptotically exact. □

Let us underline the role of the smcthness of u. In the proof of the theorem we assume that $u'''(x)$ is bounded for $x \in \widetilde{I}_i$. Instead, it could be assumed $u \in H^{2+\epsilon}(\widetilde{I}_i)$, $(\epsilon > 0)$. The elemental effectivity index ξ_i depends on the quotient $Q_i := \frac{2h_i^{\frac{1}{2}} \max_{\xi \in \widetilde{I}_i} |R'(x)|}{|e_*|_{1,I_i}}$ and the asymptotic exactness occurs if $Q_i \to 0$ as $h_i \to 0$. Nevertheless, for fixed h_i, ξ_i depends on Q_i and a high quality index requires $Q_i \ll 1$. We shall return to this problem in the subsequent sections.

Theorem 6.2. *Z-Z indicator is not asymptotically exact.*

Proof. Assume that u is a quadratic polynomial; consider a mesh such that $h_{i-1} = h_{i+1} = 2h_i$. An explicit computation shows that the elemental effectivity index is $\xi_i = 2$ independently of the meshsize h_i. Therefore Z-Z is not asymptotically exact. □

Let us now consider as a numerical example a particular problem (4.1) whose solution is given by $u(x) = \Re(x^{p+qi})$. Z-Z and B-M indicators are identical for uniform meshes, hence to observe different performances we need to use a non uniform one. Let $x_{i-2} = 0.01$, $x_{i-1} = 0.04$, $x_i = 0.09$, $x_{i+1} = 0.16$. Table 6.1 shows the elemental effectivity index ξ_i for both indicators.

	$q = 0.0$		$q = 0.5$	
p	Z-Z	B-M	Z-Z	B-M
0.55	1.07	1.78	4.70	6.19
0.65	1.03	1.68	3.18	5.21
0.75	0.99	1.57	1.59	2.80
0.85	0.97	1.48	1.18	2.07
0.95	0.96	1.39	1.01	1.69
1.05	0.96	1.32	0.94	1.45
1.15	0.96	1.26	0.92	1.29
1.25	0.97	1.20	0.93	1.17
1.35	0.98	1.15	0.97	1.08
1.45	1.01	1.11	1.01	1.02
1.55	1.03	1.07	1.07	0.99
1.65	1.07	1.04	1.13	0.97
1.75	1.10	1.02	1.20	0.96
1.85	1.15	1.01	1.27	0.97
1.95	1.19	1.00	1.35	0.99
2.05	1.24	1.00	1.44	1.02
2.15	1.30	1.01	1.52	1.06
2.25	1.35	1.02	1.61	1.11
2.35	1.41	1.04	1.71	1.16
2.45	1.48	1.06	1.81	1.21

Table 6.1. Elemental effectivity indices for different singular solutions.

We observe that the Z-Z indicator is better for nonsmooth solutions ($p < 1.5$) and the B-M indicator is better for smooth u. The inaccuracy of the B-M indicator for p small is directly related to the nonsmoothness of the function u and the poor quality of the Z-Z indicator for p large is related to the fact that it is not asymptotically exact. Table 6.1 shows that to measure the quality of an indicator is not a simple task. In the next sections we shall address in more detail the observed effects.

Let us consider now a particular problem (4.1) whose solution is given by

$$u(x) = x^p - x \ , \qquad x \in I = (0,1) \ . \tag{6.3}$$

For $p > 1.5$, $u \in H^{2+\epsilon}(I)$, $(\epsilon > 0)$, and hence the asymptotic exactness of the B-M indicator should be reflected in its performance. But for $p < 1.5$ we cannot expect to observe high quality effectivity indices for this indicator.

Remark 6.1. We study these kind of problems with solutions (6.3) since they are one dimensional models for the singularity of the solutions of two dimensional elliptic problems in the neighborhood of a corner. ☐

We consider graded meshes with nodes

$$x_i := \left(\frac{i}{n}\right)^{\beta}, \qquad i = 0, \ldots, n\,, \tag{6.4}$$

with $\beta > 0$. Table 6.2 shows the true error and the elemental effectivity indices ξ_i for $n = 10$, for $p = 1.25$ and $p = 2.25$, and for $\beta = 2.0$ and $\beta = 0.5$.

	$p = 1.25$					
	$\beta = 2.0$			$\beta = 0.5$		
Int.	error	Z-Z	B-M	error	Z-Z	B-M
1	6.45(-3)	1.172	0.391	8.61(-2)	0.256	0.618
2	7.93(-3)	0.937	1.441	8.85(-3)	3.574	1.392
3	8.02(-3)	0.968	1.198	4.86(-3)	1.254	1.034
4	8.04(-3)	0.983	1.103	3.31(-3)	1.116	1.016
5	8.05(-3)	0.990	1.063	2.49(-3)	1.068	1.009
6	8.06(-3)	0.993	1.042	1.98(-3)	1.044	1.006
7	8.06(-3)	0.995	1.030	1.64(-3)	1.032	1.004
8	8.06(-3)	0.996	1.023	1.40(-3)	1.024	1.003
9	8.06(-3)	0.997	1.018	1.21(-3)	1.018	1.002
10	8.07(-3)	0.970	1.084	1.07(-3)	1.080	1.021
	$p = 2.25$					
	$\beta = 2.0$			$\beta = 0.5$		
Int.	error	Z-Z	B-M	error	Z-Z	B-M
1	2.11(-4)	4.024	1.341	8.91(-2)	0.469	1.131
2	1.67(-3)	1.807	1.050	3.02(-2)	1.879	0.969
3	4.57(-3)	1.352	1.019	2.17(-2)	1.116	0.996
4	8.93(-3)	1.194	1.010	1.75(-2)	1.054	0.998
5	1.47(-2)	1.122	1.006	1.50(-2)	1.031	0.999
6	2.20(-2)	1.083	1.004	1.32(-2)	1.021	0.999
7	3.07(-2)	1.060	1.003	1.19(-2)	1.015	0.999
8	4.09(-2)	1.045	1.002	1.09(-2)	1.011	1.000
9	5.25(-2)	1.036	1.002	1.01(-2)	1.008	1.000
10	6.56(-2)	0.872	0.974	9.38(-3)	1.050	0.993

Table 6.2. Elemental effectivity indices for different graded meshes.

We observe the following features:

i) Z-Z indicator is of high quality when the mesh is equilibrated; i.e. the mesh is such that the errors are almost equidistributed in all the subintervals. (In the next section we shall address this feature in detail).

ii) B-M indicator is in general of higher quality, except for those meshes that are well equilibrated and when the solution is not sufficiently smooth.

iii) The performance of Z-Z and B-M indicators in the first and the last intervals is in general poor. This feature is related to extension aspects.

7. Further analysis of Z-Z and B-M indicators and their generalization. As we have seen in section 4, $U(x_i)$ is a weighted average of u'_{FE} on I_i and I_{i+1}. For both indicators, Z-Z and B-M, $U(x_i)$ lies between $u'_{FE}|_{I_i}$ and $u'_{FE}|_{I_{i+1}}$; in fact, according to (4.4), (4.11) and (4.12):

$$U(x_i) = u'_{FE}|_{I_i} + \alpha_i \left[u'_{FE}\right]_i = u'_{FE}|_{I_i} + \alpha_i \left(u'_{FE}|_{I_{i+1}} - u'_{FE}|_{I_i}\right) ,$$

with

$$\alpha_i = \alpha_i^{ZZ} := \frac{1}{1+\Delta_i} , \tag{7.1}$$

for Z-Z indicator and

$$\alpha_i = \alpha_i^{BM} := \frac{1}{1+\Delta_i^{-1}} . \tag{7.2}$$

for B-M indicator, where $\Delta_i := \frac{h_i}{h_{i+1}}$.

Let us now address the problem of how to select α_i. Using (4.7), (4.8) and the relation $\alpha_i = 1 + \beta_i$, the indicator can be written:

$$\eta_i^2 = \frac{h_i}{3}\left[(1-\alpha_{i-1})^2\left[u'_{FE}\right]_{i-1}^2 - \alpha_i(1-\alpha_{i-1})\left[u'_{FE}\right]_{i-1}\left[u'_{FE}\right]_i + \alpha_i^2\left[u'_{FE}\right]_i^2\right] . \tag{7.3}$$

For $x \in I_i$ let

$$u'(x) = u'_{FE}|_{I_i} + a_i(x - x_{i-1/2}) + r_i(x) , \tag{7.4}$$

with $x_{i-1/2} := \frac{x_{i-1}+x_i}{2}$ and

$$a_i := \frac{\int_{I_i} u'(x)(x-x_{i-1/2})dx}{\int_{I_i}(x-x_{i-1/2})^2 dx} = \frac{12}{h_i^3}\int_{I_i} e'_u(x)(x-x_{i-1/2})dx = -\frac{12}{h_i^3}\int_{I_i} e_u(x)dx ;$$

hence $r_i(x)$ is orthogonal to 1 and to $(x - x_{i-1/2})$, (that is, $\int_{I_i} r_i(x)dx = 0$ and $\int_{I_i} r_i(x)(x - x_{i-1/2})dx = 0$).

Let us assume that $u''(x) > 0$ (or $u''(x) < 0$) for all $x \in \widetilde{I}_i$. Then $e_u(x) < 0$ (resp. $e_u(x) > 0$) $\forall x \in (x_{j-1}, x_j)$, $j = i-1, i, i+1$, and so $a_{i-1}, a_i, a_{i+1} > 0$ (resp. $a_{i-1}, a_i, a_{i+1} < 0$). Let L_i and R_i be defined by

$$(7.5) \qquad \begin{aligned} e'_u|_{I_i}(x_{i-1}) &= -\frac{a_i h_i}{2} + r_i(x_{i-1}) = -\frac{a_i h_i}{2}(1 - L_i) \\ e'_u|_{I_i}(x_i) &= \frac{a_i h_i}{2} + r_i(x_i) = \frac{a_i h_i}{2}(1 + R_i) \ . \end{aligned}$$

Let K_i be defined by

$$(7.6) \qquad |e_u|^2_{1,I_i} = \frac{a_i^2 h_i^3}{12} + \int_{I_i} r_i^2(x)dx = \frac{a_i^2 h_i^3}{12}(1 + K_i^2) \ .$$

If u is sufficiently smooth, $L_i, R_i, K_i = \mathcal{O}(h_i)$.

Under the assumption $u''(x) > 0$ (or $u''(x) < 0$) $\forall x \in \widetilde{I}_i$, the jumps $\left[u'_{FE}\right]_{i-1}$ and $\left[u'_{FE}\right]_i$ are strictly positive (resp. strictly negative) and hence there exist coefficients α_{i-1} and α_i such that

$$(7.7) \qquad \begin{aligned} (1 - \alpha_{i-1})\left[u'_{FE}\right]_{i-1} &= \frac{a_i h_i}{2}\left(1 + \frac{K_i}{\sqrt{3}}\right) \\ \alpha_i \left[u'_{FE}\right]_i &= \frac{a_i h_i}{2}\left(1 - \frac{K_i}{\sqrt{3}}\right) \ . \end{aligned}$$

Using these coefficients in (7.3) we obtain for the corresponding estimator

$$\eta_i^2 = \frac{a_i^2 h_i^3}{12}(1 + K_i^2) = |e_u|^2_{1,I_i} \ ;$$

that is, the coefficients defined in (7.7) are optimal in the sense that they yield an exact estimator of the error.

By using (7.5) the optimal coefficient α_i can be written

$$(7.8) \qquad \alpha_i = \frac{\left(\frac{1+K_i/\sqrt{3}}{1+R_i}\right)}{1 + \left(\frac{1-L_{i+1}}{1+R_i}\right)\left(\frac{a_{i+1}h_{i+1}}{a_i h_i}\right)}$$

and using (7.6), since a_i and a_{i+1} have the same sign,

$$(7.9) \qquad \alpha_i = \frac{\left(\frac{1+K_i/\sqrt{3}}{1+R_i}\right)}{1 + \left(\frac{1-L_{i+1}}{1+R_i}\right)\left(\frac{1+K_{i+1}^2}{1+K_i^2}\right)^{\frac{1}{2}}\left(\frac{|e_u|_{1,I_{i+1}}}{|e_u|_{1,I_i}}\right)\Delta_i^{\frac{1}{2}}} \ .$$

A similar expression holds for the other optimal coefficient α_{i-1}.

Therefore, whenever the mesh is equilibrated with respect to the energy norm (i.e. $|e_u|_{1,I_{i-1}} \approx |e_u|_{1,I_i} \approx |e_u|_{1,I_{i+1}}$), if u is smooth enough as $R_{i-1}, L_i, R_i, L_{i+1}, K_i = o(1)$ to hold, then the optimal coefficients are

$$\alpha_{i-1} = \frac{1}{1+\Delta_{i-1}^{\frac{1}{2}}} + o(1) \qquad \text{and} \qquad \alpha_i = \frac{1}{1+\Delta_i^{\frac{1}{2}}} + o(1)$$

and we have the following theorem.

THEOREM 7.1. *Let $u''(x) > 0$ (or $u''(x) < 0$) for all $x \in I$ and let $R_i, L_i, K_i = o(1)$, $i = 1, \ldots, n$. Then, if the mesh is equilibrated with respect to the energy norm, the coefficients*

$$\alpha_i^{\text{opt}} := \frac{1}{1+\Delta_i^{\frac{1}{2}}} \tag{7.10}$$

yield an asymptotically exact estimator.

The values of L_i, R_i and K_i govern the effectivity of the indicators. If the solution u is less smooth, the values of these constants are larger and we have to expect less accurate effectivity indices. We shall show that when the solution has a very strong singularity, the estimator defined by α_i^{opt} is not necessarily optimal.

Consider the model problem of section 6 with solution (6.3). In this case $u''(x)$ does not change sign in I and so Theorem 7.1 can be applied. However for $\frac{1}{2} < p < 1$, L_i, R_i and K_i are not negligible and become larger as $p \to \frac{1}{2}$. In fact, in this particular case, when using meshes equilibrated with respect to the energy norm, we shall see that choosing

$$\alpha_i^{\gamma} := \frac{1}{1+\Delta_i^{\gamma}}$$

with $\gamma > \frac{1}{2}$ can be preferable.

In Table 7.1 we show the results for various values of p and the corresponding values of β in (6.4) leading to asymptotically equilibrated meshes.

	$p = 0.75$ – $\beta = 6$			$p = 1.25$ – $\beta = 2$		
Int.	error	$\gamma = 1$	$\gamma = \frac{1}{2}$	error	$\gamma = 1$	$\gamma = \frac{1}{2}$
1	1.12(-2)	1.05564	0.95241	6.45(-3)	1.17157	0.99033
2	2.12(-2)	1.00429	0.75513	7.93(-3)	0.93675	0.90257
3	2.35(-2)	0.99922	0.86071	8.02(-3)	0.96772	0.97650
4	2.43(-2)	0.98554	0.92311	8.04(-3)	0.98300	0.98921
5	2.46(-2)	0.98309	0.95261	8.05(-3)	0.98962	0.99374
6	2.48(-2)	0.98470	0.96809	8.06(-3)	0.99302	0.99590
7	2.49(-2)	0.98708	0.97710	8.06(-3)	0.99499	0.99710
8	2.50(-2)	0.98925	0.98278	8.06(-3)	0.99623	0.99784
9	2.50(-2)	0.99106	0.98659	8.06(-3)	0.99707	0.99833
10	2.50(-2)	0.83594	0.98765	8.07(-3)	0.96997	0.99848
	$p = 1.75$ – $\beta = \frac{6}{5}$			$p = 2.25$ – $\beta = \frac{6}{7}$		
Int.	error	$\gamma = 1$	$\gamma = \frac{1}{2}$	error	$\gamma = 1$	$\gamma = \frac{1}{2}$
1	1.50(-2)	1.05414	0.99399	2.11(-2)	0.95182	1.00696
2	1.57(-2)	0.95943	0.97301	2.04(-2)	1.04344	1.02352
3	1.57(-2)	0.99164	0.99574	2.04(-2)	1.00638	1.00288
4	1.57(-2)	0.99608	0.99809	2.04(-2)	1.00292	1.00128
5	1.57(-2)	0.99770	0.99890	2.04(-2)	1.00169	1.00074
6	1.57(-2)	0.99848	0.99928	2.04(-2)	1.00111	1.00048
7	1.57(-2)	0.99892	0.99949	2.04(-2)	1.00079	1.00034
8	1.57(-2)	0.99920	0.99962	2.04(-2)	1.00059	1.00025
9	1.57(-2)	0.99938	0.99971	2.04(-2)	1.00046	1.00020
10	1.57(-2)	0.99414	0.99974	2.04(-2)	1.00414	1.00018

Table 7.1. Comparison of indicators for equilibrated meshes with respect to the energy norm.

This table shows that for $p = .75$, $\gamma = 1$ (i.e. Z-Z indicator) yields better effectivity indices than $\gamma = \frac{1}{2}$. The particular good performance of this estimator is related with the monotonicity of u and is not always valid when the solution is highly singular (see Table 6.1). For $p = 0.75$ and for $p = 1.25$, the effectivity index in the first elements of the estimator corresponding to $\gamma = \frac{1}{2}$ is not so close to one, because the mesh is not so well equilibrated.

Remark 7.1. Assume once more that u is sufficiently smooth for L_i, R_i and K_i to be negligible in (7.5) and (7.6). Then the optimal coefficients (7.8) are

$$\alpha_i \approx \frac{1}{1 + \frac{a_{i+1} h_{i+1}}{a_i h_i}} \,. \tag{7.11}$$

Let $\|e'_u\|_{L^s(I_i)} := \left[\int_{I_i} |e'_u(x)|^s dx\right]^{\frac{1}{s}}$ be the usual $L^s(I_i)$–norm of e'_u. A simple Taylor series argument applied to (7.4) shows that if u is smooth enough then

$$\|e'_u\|_{L^s(I_i)} \approx \frac{2|a_i|h_i^{1+\frac{1}{s}}}{(s+1)^{\frac{1}{s}}}.$$

If the mesh is equilibrated with respect to this norm, then the optimal coefficients are

$$\alpha_i \approx \alpha_i^{(s)} := \frac{1}{1+\Delta_i^{\frac{1}{s}}} \tag{7.12}$$

up to higher order terms. Obviously (7.10) is a particular case of this formula with $s = 2$. But for instance, if the mesh is equilibrated with respect to the norm $\|e'_u\|_{L^1(I_i)}$, then Z-Z indicator is the optimal. On the other hand, when $s \to \infty$, $\alpha_i^{(s)} \to \frac{1}{2}$; that is, the simple average of $u'_{FE}|_{I_i}$ and $u'_{FE}|_{I_{i+1}}$ is the correct value of $U(x_i)$ when the mesh is equilibrated with respect to the norm $\|e'_u\|_{L^\infty(I_i)}$. □

If we knew the exact errors $|e_u|_{1,I_i}$ for a given mesh, then we could use them in (7.9) to obtain optimal coefficients

$$\widehat{\alpha}_i^{(2)} := \frac{1}{1+\left(\frac{|e_u|_{1,I_{i+1}}}{|e_u|_{1,I_i}}\right)\Delta_i^{\frac{1}{2}}}. \tag{7.13}$$

Using this information, we could improve the effectivity indices. We show this effect in the following table.

	$p = 0.75$ – $\beta = 9.0$				$p = 1.75$ – $\beta = 2.5$			
Int.	error	$\alpha_i^{(1)}$	$\alpha_i^{(2)}$	$\widehat{\alpha}_i^{(2)}$	error	$\alpha_i^{(1)}$	$\alpha_i^{(2)}$	$\widehat{\alpha}_i^{(2)}$
1	1.99(-3)	1.290	1.237	1.095	3.56(-4)	3.247	2.696	1.036
2	8.11(-3)	1.118	0.881	1.011	2.15(-3)	1.656	1.373	1.022
3	1.42(-2)	1.128	0.861	0.907	4.96(-3)	1.286	1.146	1.005
4	1.95(-2)	1.110	0.897	0.929	8.59(-3)	1.156	1.076	1.002
5	2.42(-2)	1.084	0.928	0.950	1.29(-2)	1.097	1.047	1.001
6	2.86(-2)	1.063	0.948	0.964	1.79(-2)	1.066	1.031	1.001
7	3.27(-2)	1.048	0.961	0.973	2.35(-2)	1.048	1.023	1.001
8	3.66(-2)	1.037	0.970	0.979	2.97(-2)	1.036	1.017	1.000
9	4.04(-2)	1.029	0.976	0.984	3.64(-2)	1.028	1.013	1.000
10	4.40(-2)	0.700	0.936	0.986	4.36(-2)	0.875	0.914	1.000

Table 7.2

Obviously the exact errors $|e_u|_{1,I_i}$ are not available. Nevertheless, if we use the estimated errors and if their ratio is reasonably close to the ratio of the true errors, then it is possible to iterate. Table 7.3 shows the results obtained by this iterative procedure.

	$p = 0.75$ – $\beta = 9.0$				$p = 1.75$ – $\beta = 2.5$			
Int.	error	$\alpha_i^{(2)}$	Iter.	$\widehat{\alpha}_i^{(2)}$	error	$\alpha_i^{(2)}$	Iter.	$\widehat{\alpha}_i^{(2)}$
1	1.99(-3)	1.237	1.131	1.095	3.56(-4)	2.696	1.242	1.036
2	8.11(-3)	0.881	0.895	1.011	2.15(-3)	1.373	0.963	1.022
3	1.42(-2)	0.861	0.909	0.907	4.96(-3)	1.146	1.034	1.005
4	1.95(-2)	0.897	0.934	0.929	8.59(-3)	1.076	0.980	1.002
5	2.42(-2)	0.928	0.957	0.950	1.29(-2)	1.047	1.019	1.001
6	2.86(-2)	0.948	0.962	0.964	1.79(-2)	1.031	0.986	1.001
7	3.27(-2)	0.961	0.981	0.973	2.35(-2)	1.023	1.013	1.001
8	3.66(-2)	0.970	0.971	0.979	2.97(-2)	1.017	0.989	1.000
9	4.04(-2)	0.976	0.998	0.984	3.64(-2)	1.013	1.010	1.000
10	4.40(-2)	0.936	0.968	0.986	4.36(-2)	0.914	0.991	1.000

Table 7.3

As a conclusion, let us remark that for an adaptive procedure based on equilibrated meshes, it is convenient to use the proper optimal indicator. The Z-Z indicator is not the proper one when the adaptive procedure is based on equilibration with respect to the energy norm. Nevertheless it is reasonably close, but the proper one is preferable. The estimates could be improved if a-posteriori information about the errors is used, namely the quotient of computed estimated errors. The results for the first and last intervals are less reliable because of extension aspects.

8. Robustness and quality measure of an indicator. In section 4 we introduced a family of indicators based on coefficients α_i, β_{i-1} (see (4.7-8)) and in the subsequent sections we discussed their performance from an asymptotic point of view. This approach cannot be well applied when the asymptotic assumptions (i.e. L_i, R_i, $K_i = o(1)$ as in section 7) are not satisfied.

In (5.4) we introduced another measure of the quality of an indicator:

$$\|\mathcal{L}_i\| := \sup_{v\in\mathcal{H}} \frac{|e'_v - V_i v|_{0,I_i}}{\|v\|_{\mathcal{H}}},$$

that can be used when the solution is known to belong to a subspace $\mathcal{H} \subset H_0^1(I)$. Now, based on this measure, we shall develop principles for judging an indicator without asymptotic assumptions. (From now on, we write $V_i u$, $V_i v$ instead of V_i in (4.7), since we shall need to deal with functions V_i defined by solutions u, v of different problems).

Since the indicators defined by (4.7-8) satisfy $|e'_v - V_i v|_{0,I_i} = 0$ when v is a linear function in $\widetilde{I}_i$, then we can analyze the space $\mathcal{H}$ modulo $\mathcal{P}_1$ (the space of linear functions). One of the possibilities is to consider $H^2(\widetilde{I}_i)/\mathcal{P}_1$; however, let us note that $\|\mathcal{L}_i\|$ is well defined even for $H^1(\widetilde{I}_i)/\mathcal{P}_1$.

On the other hand, $\mathcal{L}_i$ depends on the particular indicator (i.e. on α_i and β_{i-1} in (4.7)), on the mesh $\mathcal{T}$ and on the particular space $\mathcal{H}$. We shall write explicitly this dependence; so, instead of (5.4), let us call

$$(8.1) \qquad \mathcal{R}(\mathcal{H},\alpha,\beta,\mathcal{T}) := \sup_{v\in\mathcal{H}/\mathcal{P}_1} \frac{|e'_v - V_i v|_{0,I_i}}{\|v\|_{\mathcal{H}}} = \sup_{v\in\mathcal{H}/\mathcal{P}_1} \frac{|e'_v - V_i v|_{0,I_i}}{\|v\|_{\mathcal{H}/\mathcal{P}_1}},$$

the *robustness index* of the indicator; (from now on we omit the subindices of the coefficients α_i and β_{i-1} defining $V_i v$).

If we do not know anything more about the solution u than that $u \in \mathcal{H}$, then the safest choice of the indicator is one whose robustness index is minimal. In fact, for any pair (α,β) there exists $\widehat{u} \in \mathcal{H}$ such that

$$\mathcal{R}(\mathcal{H},\alpha,\beta,\mathcal{T}) = \frac{|e'_{\widehat{u}} - V_i\widehat{u}|_{0,I_i}}{\|\widehat{u}\|_{\mathcal{H}}}$$

and so we accept the possibility that $\widehat{u}$ can be the solution of our problem. Nevertheless, $\widehat{u}$ (which depends on $\mathcal{H}$) could be an *unlikely candidate as the solution* of the problem under consideration. This would indicate that the space $\mathcal{H}$ is too large for our purpose. So, the role of $\mathcal{H}$ for the quality assesment is obvious.

Sometimes, it is known a-priori that the solution u belongs to all the spaces of a certain family. To deal with this case, we introduce some notation. Let $\mathcal{K}_{\Gamma,\psi}$ denote a one–parameter family of spaces $\mathcal{H}_\gamma$, $\gamma \in \Gamma$, where $\psi : \Gamma \to \mathbf{R}$ is a positive function; we say that $u \in \mathcal{K}_{\Gamma,\psi}$ if $u \in \mathcal{H}_\gamma$, $\forall\gamma \in \Gamma$ and

$$(8.2) \qquad \exists C > 0 : \quad \psi(\gamma)\|u\|_{\mathcal{H}_\gamma} \le C, \quad \forall\gamma \in \Gamma .$$

For example, if $\Gamma = \mathbf{N}$, $\mathcal{H}_n = H^n(I)$ and $\psi(n) = n!e^{-n}$, $n \in \mathbf{N}$, then (8.2) holds if and only if u is analytic in I; in [2,3], these kind of spaces have been used for the characterization of the regularity of the solutions of elliptic partial differential equations in the neighborhood of a corner of the domain.

If for a given problem, we know not only that $u \in \mathcal{K}_{\Gamma,\psi}$, but also $\|u\|_{\mathcal{H}_\gamma/\mathcal{P}_1}$ for $\gamma \in \Gamma$, we could then compute

$$(8.3) \qquad \mathcal{Q}(\mathcal{K}_{\Gamma,\psi},\alpha,\beta,\mathcal{T},u) := \inf_{\gamma\in\Gamma} \left\{ \mathcal{R}(\mathcal{H}_\gamma,\alpha,\beta,\mathcal{T})\|u\|_{\mathcal{H}_\gamma/\mathcal{P}_1} \right\}$$

and since $|e'_v - V_i v|_{0,I_i} \le \mathcal{R}(\mathcal{H}_\gamma,\alpha,\beta,\mathcal{T})\|u\|_{\mathcal{H}_\gamma/\mathcal{P}_1}$ for all $\gamma \in \Gamma$, $\mathcal{Q}(\mathcal{K}_{\Gamma,\psi},\alpha,\beta,\mathcal{T},u)$ could then be used to asses the quality of the indicator for this particular problem.

In practice, we may know $\|u\|_{\mathcal{H}_\gamma/\mathcal{P}_1}$ only approximately. If we denote $\|u\|^*_{\mathcal{H}_\gamma/\mathcal{P}_1}$ the approximation of $\|u\|_{\mathcal{H}_\gamma/\mathcal{P}_1}$ obtained by using the (a-posteriori) information that we have about the solution, then we call

$$\mathcal{Q}^*(\mathcal{K}_{\Gamma,\psi},\alpha,\beta,\mathcal{T},u) := \inf_{\gamma\in\Gamma}\left\{\mathcal{R}(\mathcal{H}_\gamma,\alpha,\beta,\mathcal{T})\|u\|^*_{\mathcal{H}_\gamma/\mathcal{P}_1}\right\} \tag{8.4}$$

the *quality index* of the indicator for this particular problem.

9. Computation of robustness and quality indices. In this section we shall address as an example the computation of the indices defined above for some weighted Sobolev spaces modeling the space of solutions of elliptic PDEs in the neigborhood of corners.

Let $\widetilde{I}$ be an interval and $m \geq l \geq 0$ be integers, then by $H^{m,l}_\gamma(\widetilde{I})$ we denote the weighted Sobolev space that is the completion of the set of the infinitely differentiable functions under the norm

$$\|v\|^{(m,l)}_{\gamma,\widetilde{I}} := \left[\sum_{k=0}^{l-1}\|v^{(k)}(x)\|^2_{0,\widetilde{I}} + \sum_{k=l}^{m}\|x^{\gamma+k-l}\, v^{(k)}(x)\|^2_{0,\widetilde{I}}\right]^{1/2},$$

where $\sum_{k=0}^{-1}$ is void and $v^{(k)} := \frac{d^kv}{dx^k}$. For $k = l,\ldots,m$, let

$$|v|^{(k,l)}_{\gamma,\widetilde{I}} := \|x^{\gamma+k-l}\, v^{(k)}(x)\|_{0,\widetilde{I}}\,.$$

Let us introduce first some auxiliary theoretical results.

LEMMA 9.1. *If $0 < \gamma \leq \gamma' < 1$ and $m \geq l \geq 0$, then the inclusion $H^{m+1,l}_\gamma(\widetilde{I}) \hookrightarrow H^{m,l}_{\gamma'}(\widetilde{I})$ is compact.*

Proof. See [21] (page 287). □

THEOREM 9.1. *Let $(\alpha,\beta) \in \mathbf{R}^2$ and $\gamma \in (0,1)$, then*

$$\|e'_v - V_i v\|_{0,I_i} \leq C(\alpha,\beta,\gamma,\mathcal{T})\|v\|_{H^{2,2}_\gamma(\widetilde{I}_i)/\mathcal{P}_1}\,; \tag{9.1}$$

i.e. $(e'_v - V_i v)$ is a bounded linear operator of $H^{(2,2)}_\gamma(\widetilde{I}_i)/\mathcal{P}_1$ into $L^2(I_i)$.

Proof. Since the inclusion $H^{2,2}(\widetilde{I}) \hookrightarrow H^1(\widetilde{I})$ is continuous, then (9.1) holds when on the right hand side $\|v\|_{H^{2,2}_\gamma(\widetilde{I}_i)}$ is used instead of $\|v\|_{H^{2,2}_\gamma(\widetilde{I}_i)/\mathcal{P}_1}$. Since $e'_v - V_i v = 0$ for $v \in \mathcal{P}_1$, (9.1) follows immediately. □

So we have:

COROLLARY 9.1. *The robustness index is defined for any* $(\alpha, \beta) \in \mathbf{R}^2$, *any mesh* $\mathcal{T}$ *and* $\mathcal{H} = H_\gamma^{2,2}(\widetilde{I}_i)$, *with* $0 < \gamma < 1$.

For $0 < \gamma < 1$, let

$$(v, w)_{\gamma,i}^{3,2} := \int_{\widetilde{I}_i} \left[x^{2\gamma+2} v'''(x) w'''(x) + x^{2\gamma} v''(x) w''(x) \right] dx$$

denote the scalar product in the Hilbert space $H_\gamma^{3,2}(\widetilde{I}_i)/\mathcal{P}_1$ and $|\cdot|_{\gamma,i}^{3,2} := \left[(\cdot,\cdot)_{\gamma,i}^{3,2} \right]^{\frac{1}{2}}$ the corresponding norm.

For any $v \in H_\gamma^{3,2}(\widetilde{I}_i)/\mathcal{P}_1$, there exists a unique $z_v \in H_\gamma^{3,2}(\widetilde{I}_i)/\mathcal{P}_1$ such that

$$(z_v, w)_{\gamma,i}^{3,2} = \int_{I_i} (e_v' - V_i v)(e_w' - V_i w) dx \ , \quad \forall w \in H_\gamma^{3,2}(\widetilde{I}_i)/P_1 \ . \tag{9.2}$$

(this follows from Theorem 9.1). Hence, $T(v) := z_v$ defines an operator of $H_\gamma^{3,2}(\widetilde{I}_i)/\mathcal{P}_1$ into itself and we have the following theorem.

THEOREM 9.2. *T is a self–adjoint, positive definite, compact, linear operator of $H_\gamma^{3,2}(\widetilde{I}_i)/\mathcal{P}_1$ into itself.*

Proof. It is a simple consequence of the definition and Lemma 9.1. $\square$

Thus we have:

COROLLARY 9.2.

$$\mathcal{R}(H_\gamma^{3,2}(\widetilde{I}_i), \alpha, \beta, \mathcal{T}) = \lambda^{\frac{1}{2}} \ ,$$

where λ is the maximal eigenvalue of the problem $T(v) = \lambda v$.

Let us now consider the mesh $\mathcal{T}$ with nodes $x_i := \left(\frac{i}{10}\right)^2$, $(I_i := (x_{i-1}, x_i))$. In Table 9.1 we show $\mathcal{R}(H_\gamma^{3,2}(\widetilde{I}_i), \alpha, \beta, \mathcal{T})$ for $\gamma = 0.0$ and $\gamma = 0.9$ at each interior subinterval, for Z-Z and B-M indicators and for the optimal indicator in the sense of minimising $\mathcal{R}$. $\gamma = 0.0$ corresponds to a space of smooth functions and $\gamma = 0.9$ to a set of functions with a strong singularity at the origin.

We observe that for very nonsmooth functions ($\gamma = 0.9$) the robustness index indicates that the Z-Z indicator performs better than the B-M indicator on those intervals closest to the singularity ($i = 2, 3, 4$). On the other hand, for smooth functions ($\gamma = 0.0$) B-M robustness indices are always better.

Now let $\mathcal{K}_{\Gamma,\psi}$ be the one–parameter family of spaces $H_\gamma^{3,2}(\widetilde{I}_2)$, $\gamma \in \Gamma := (0,1)$, with $\psi(\gamma) = 1, \forall \gamma \in \Gamma$, $(I_2 = (0.01, 0.04)$ and $\widetilde{I}_2 = (0.00, 0.09))$. In Table 9.2 we show the quality indices $\mathcal{Q}(\mathcal{K}_{\Gamma,\psi}, \alpha, \beta, \mathcal{T}, u)$ for problems which solutions are the functions $u(x) = \Re(x^{p+qi})$ for various values of p and $q = 0.0$ and $q = 0.5$. We use the exact value of $\|u\|_{H_\gamma^{3,2}(\widetilde{I}_2)/\mathcal{P}_1}$ to compute Q. We also show in this table, the exact error $|e'_u|_{0,I_2}$, the estimated error $|V_i u|_{0,I_2}$, the effectivity index ξ_2, the error in the estimate of e'_u (i.e. err. est.:= $|e'_u - V_i u|_{0,I_2}$) and the percentage of this last error (i.e. %:= $100\frac{|e'_u - V_i u|_{0,I_2}}{|e'_u|_{0,I_2}}$).

	$\gamma = 0.0$			$\gamma = 0.9$		
Int.	Z-Z	B-M	optimal	Z-Z	B-M	optimal
2	0.451(-4)	0.232(-4)	0.103(-4)	0.139(-1)	0.645(-1)	0.897(-2)
3	0.630(-4)	0.363(-4)	0.165(-4)	0.542(-2)	0.865(-2)	0.254(-2)
4	0.731(-4)	0.440(-4)	0.229(-4)	0.227(-2)	0.244(-2)	0.983(-3)
5	0.786(-4)	0.484(-4)	0.274(-4)	0.110(-2)	0.978(-3)	0.481(-3)
6	0.819(-4)	0.509(-4)	0.307(-4)	0.595(-3)	0.476(-3)	0.269(-3)
7	0.839(-4)	0.525(-4)	0.336(-4)	0.348(-3)	0.262(-3)	0.160(-3)
8	0.852(-4)	0.536(-4)	0.363(-4)	0.218(-3)	0.157(-3)	0.102(-3)
9	0.861(-4)	0.543(-4)	0.378(-4)	0.143(-3)	0.101(-3)	0.685(-4)

Table 9.1. Robustness indices.

$q = 0.0$		Z-Z				
p	$\lvert e'_u\rvert_{0,I_2}$	$\lvert V_i u\rvert_{0,I_2}$	ξ_2	err. est.	%	Q
0.55	0.94(-1)	0.11(-0)	1.14	0.52(-1)	55.6	0.21(-0)
0.75	0.33(-1)	0.29(-1)	0.89	0.84(-2)	25.8	0.40(-1)
0.95	0.38(-2)	0.32(-2)	0.84	0.11(-2)	28.6	0.42(-2)
1.15	0.64(-2)	0.57(-2)	0.89	0.28(-2)	43.1	0.66(-2)
1.35	0.82(-2)	0.81(-2)	0.99	0.48(-2)	58.7	0.84(-2)
1.55	0.69(-2)	0.78(-2)	1.13	0.52(-2)	74.8	0.73(-2)
1.75	0.50(-2)	0.65(-2)	1.29	0.46(-2)	91.8	0.56(-2)
1.95	0.33(-2)	0.49(-2)	1.48	0.37(-2)	110.5	0.42(-2)
2.15	0.21(-2)	0.36(-2)	1.69	0.28(-2)	131.3	0.31(-2)
2.35	0.13(-2)	0.25(-2)	1.93	0.20(-2)	154.8	0.22(-2)

Table 9.2.a. Quality indices Q of Z-Z indicator for $u(x) = \Re(x^{p+qi})$, ($q = 0.0$).

$q = 0.0$		B-M				
p	$\lvert e'_u\rvert_{0,I_2}$	$\lvert V_i u\rvert_{0,I_2}$	ξ_2	err. est.	%	Q
0.55	0.94(-1)	0.35(-0)	3.73	0.31(-0)	326.9	0.50(-0)
0.75	0.33(-1)	0.86(-1)	2.63	0.70(-1)	215.9	0.95(-1)
0.95	0.38(-2)	0.76(-2)	2.00	0.57(-2)	150.5	0.84(-2)
1.15	0.64(-2)	0.10(-1)	1.59	0.68(-2)	106.5	0.11(-1)
1.35	0.82(-2)	0.11(-1)	1.32	0.61(-2)	73.9	0.11(-1)
1.55	0.69(-2)	0.79(-2)	1.14	0.33(-2)	47.8	0.79(-2)
1.75	0.50(-2)	0.52(-2)	1.04	0.13(-2)	25.4	0.50(-2)
1.95	0.33(-2)	0.33(-2)	1.00	0.17(-3)	4.9	0.32(-2)
2.15	0.21(-2)	0.22(-2)	1.02	0.31(-3)	14.7	0.22(-2)
2.35	0.13(-2)	0.14(-2)	1.09	0.45(-3)	34.4	0.16(-2)

Table 9.2.b. Quality indices Q of B-M indicator for $u(x) = \Re(x^{p+qi})$, $(q = 0.0)$.

$q = 0.5$		Z-Z				
p	$\lvert e'_u\rvert_{0,I_2}$	$\lvert V_i u\rvert_{0,I_2}$	ξ_2	err. est.	%	Q
0.55	0.88(-1)	0.16(-0)	1.80	0.13(-0)	146.5	0.31(-0)
0.75	0.67(-1)	0.59(-1)	0.88	0.32(-1)	47.0	0.85(-1)
0.95	0.42(-1)	0.31(-1)	0.74	0.15(-1)	35.9	0.43(-1)
1.15	0.23(-1)	0.21(-1)	0.90	0.14(-1)	58.7	0.24(-1)
1.35	0.13(-1)	0.15(-1)	1.17	0.11(-1)	86.6	0.14(-1)
1.55	0.65(-2)	0.98(-2)	1.51	0.78(-2)	119.1	0.83(-2)
1.75	0.33(-2)	0.64(-2)	1.92	0.53(-2)	158.6	0.55(-2)
1.95	0.17(-2)	0.41(-2)	2.43	0.35(-2)	207.5	0.38(-2)
2.15	0.83(-3)	0.25(-2)	3.05	0.22(-2)	268.7	0.27(-2)
2.35	0.42(-3)	0.16(-2)	3.83	0.14(-2)	345.2	0.20(-2)

Table 9.2.c. Quality indices Q of Z-Z indicator for $u(x) = \Re(x^{p+qi})$, $(q = 0.5)$.

$q = 0.5$		B-M				
p	$\lvert e'_u\rvert_{0,I_2}$	$\lvert V_i u\rvert_{0,I_2}$	ξ_2	err. est.	%	Q
0.55	0.88(-1)	0.51(-0)	5.74	0.47(-0)	533.4	0.74(-0)
0.75	0.67(-1)	0.19(-0)	2.84	0.16(-0)	242.0	0.20(-0)
0.95	0.42(-1)	0.72(-1)	1.72	0.53(-1)	126.6	0.82(-1)
1.15	0.23(-1)	0.27(-1)	1.15	0.15(-1)	62.2	0.35(-1)
1.35	0.13(-1)	0.11(-1)	0.88	0.27(-2)	21.2	0.17(-1)
1.55	0.65(-2)	0.57(-2)	0.87	0.18(-2)	27.3	0.84(-2)
1.75	0.33(-2)	0.35(-2)	1.04	0.20(-2)	60.1	0.46(-2)
1.95	0.17(-2)	0.22(-2)	1.34	0.16(-2)	96.8	0.28(-2)
2.15	0.83(-3)	0.14(-2)	1.73	0.12(-2)	138.7	0.19(-2)
2.35	0.42(-3)	0.92(-3)	2.22	0.78(-3)	188.4	0.14(-2)

Table 9.2.d. Quality indices Q of B-M indicator for $u(x) = \Re(x^{p+qi})$, $(q = 0.5)$.

Since, in practice, the exact norm of the solution is not known, we need to estimate it. The second derivative can be recovered from the error estimate and the third one by using differences of these estimates on neighboring elements. By this way we can compute the quality indices Q^*. In Table 9.3 we report the values of Q^* corresponding to tables 9.2.

	$q = 0.0$		$q = 0.5$	
p	Z-Z	B-M	Z-Z	B-M
0.55	0.12(-0)	0.19(-0)	0.13(-0)	0.25(-0)
0.75	0.38(-1)	0.51(-1)	0.76(-1)	0.11(-0)
0.95	0.42(-2)	0.49(-2)	0.43(-1)	0.47(-1)
1.15	0.68(-2)	0.71(-2)	0.23(-1)	0.21(-1)
1.35	0.87(-2)	0.81(-2)	0.13(-1)	0.99(-2)
1.55	0.75(-2)	0.64(-2)	0.76(-2)	0.55(-2)
1.75	0.58(-2)	0.45(-2)	0.51(-2)	0.37(-2)
1.95	0.43(-2)	0.32(-2)	0.41(-2)	0.29(-2)
2.15	0.31(-2)	0.22(-2)	0.37(-2)	0.27(-2)
2.35	0.23(-2)	0.17(-2)	0.36(-2)	0.26(-2)

Table 9.3. Quality indices Q^*.

From tables 9.2 and 9.3 we observe the following features.

1) Sometimes, although $V_i u$ *does not* approximate the error e'_u well, $\|V_i u\|_{0,I_i}$ can be a close approximation of $\|e'_u\|_{0,I_i}$.
2) Z-Z indicator performs better than B-M when the solution is nonsmooth (provided that the mesh is properly graded) and B-M performs better if the solution is

smooth.

3) The quality indices correctly indicate that Z-Z indicator is preferable for nonsmooth solutions while B-M is preferable for smooth ones. In principle, a criterion for selecting Z-Z, B-M or any other indicator could be based on comparing their quality indices.

10. Conclusions. In the one-dimensional setting we have shown:

a) From an asymptotic point of view, an optimal indicator can be derived for any adaptively constructed mesh. The Z-Z indicator is optimal when the adaptive procedure equilibrates the elemental error e'_u in the L^1-norm and is not optimal if the energy norm (i.e. L^2-norm of e'_u) is equilibrated.

b) Z-Z indicator is not asymptotically exact for a general mesh while B-M is.

c) In a nonasymptotic sense, when the solution is nonsmooth and the mesh is reasonably graded, Z-Z indicator gives better results than B-M. Otherwise B-M does.

d) The quality and robustness indices can be used to select an indicator.

In the two- dimensional setting we have shown:

a) For general meshes, an asymptotically exact estimator can not be achieved by a simple L^2-projection technique.

b) For translation invariant meshes, asymptotically exact estimators can be defined by utilizing superconvergence techniques based on differences.

c) For more general meshes, the way to define a reasonable estimator is very likely to assume that the mesh is locally translation invariant (although it is not).

d) The major principles analyzed in the one-dimensional setting could be utilized in two dimensions, although further research work is needed in this direction.

REFERENCES

[1] M. Ainsworth and A. W. Craig, *A posteriori error estimators in he finite element method*, Numer. Math. (to appear).

[2] I. Babuška and B. Q. Guo, *Regularity of the solution of elliptic problems with piecewise analytical data. Part I: Boundary value problems for linear elliptic equation of second order*, SIAM J. Math. Anal., 19 (1988), pp. 172–203.

[3] I. Babuška and B. Q. Guo, *Regularity of the solution of elliptic problems with piecewise analytical data. Part II: The trace spaces and application to the boundary value problems with nonhomogeneous boundary conditions*, SIAM J. Math. Anal., 20 (1989), pp. 763–781.

[4] I. Babuška and A. Miller, *A feedback finite element method with a posteriori error estimation: Part I. The finite element method and some basic properties of the a posteriori error estimator*, Comp. Methods Appl. Mech. Engrg., 61 (1987), pp. 1–40.

[5] I. Babuška and W. C. Rheinboldt, *A posteriori error estimators in the finite element method*, Internat. J. Numer. Meth. Eng., 12 (1978), pp. 1597–1615.

[6] I. Babuška and W. C. Rheinboldt, *Error estimates for adaptive finite element computations*, SIAM J. Numer. Anal., 15 (1978), pp. 736–754.

[7] I. Babuška and W. C. Rheinboldt, *Reliable error estimation and mesh adaptation for the finite element method*, in Computational Methods in Nonlinear Mechanics (J. T. Oden, ed.), North Holland, 1980, pp. 67–108.

[8] P. L. Baehmann and M. S. Shephard, *Adaptive multiple level h-refinement in automated finite element analysis*, Eng. with Comp., 5 (1989), pp. 235–247.

[9] P. L. Baehmann, M. S. Shephard and J. E. Flaherty, *A posteriori error estimation for triangular and tetrahedral quadratic elements using interior residuals*, SCOREC report 14-1990, Rensselaer Polytechnic Institute, Troy, New York.

[10] R. E. Bank, *PLTMG. A software package for solving elliptic partial differential equations. Users guide 6.0*, SIAM, Philadelphia, 1990.

[11] R. E. Bank and A. Weiser, *Some a posteriori error estimators for elliptic partial differential equations*, Math. Comp., 44 (1985), pp. 283–301.

[12] R. E. Bank and B. D. Welfert, *A posteriori error estimators for the Stokes equation: a comparison*, in Reliability in Computational Mechanics (J. T. Oden, ed.), Elsevier Science Publisher B. V. (North Holland), 1990, pp. 323–340.

[13] J. H. Bramble and A. H. Schatz, *Higher order local accuracy by averaging in the finite element method*, Math. Comp., 31 (1977), pp. 94–111.

[14] P. Beckers and H. Zhong, *Error estimators using solution defaults and their roles in the adaptation of finite element meshes*, in FEM in the Design Process (J. Robinson, ed.), Robinson and associates, 1990, pp. 177–186.

[15] L. Demkowitz and J. T. Oden, *Extraction methods for second derivatives in finite element approximations of linear elasticity problems*, Commun. Appl. Numer. Methods, 1 (1985), pp. 137–139.

[16] L. Demkowitz, J. T. Oden, W. Rachowitz and O. Hardy, *Toward a universal h-p adaptive finite element strategy, Part 1. Constrained approximation and data structure.*, Comp. Methods Appl. Mech. Engrg., 77 (1989), pp. 79–112.

[17] R. Durán, M. A. Muschietti and R. Rodríguez, *Asymptotically exact error estimators for rectangular finite elements*, SIAM J. Numer. Anal. (to appear).

[18] R. Durán, M. A. Muschietti and R. Rodríguez, *On the asymptotic exactness of error estimators for linear triangular finite elements*, Numer. Math., 59 (1991), pp. 107–127.

[19] R. E. Ewing, *A posteriori error estimation*, in Reliability in Computational Mechanics (J. T. Oden, ed.), Elsevier Science Publisher B. V. (North Holland), 1990, pp. 59–72.

[20] C. Mesztenyi and W. Szymczak, *FEARS user's manual for UNIVAC 1100*, Tech. Note BN-991, IPST, University of Maryland, 1982.

[21] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, Editions de l'Académie, Prague, 1967.

[22] J. T. Oden, *Adaptive finite element methods for problems in solid and fluid mechanics*, in Finite Elements, Theory and Application (D. L. Dwoyer, M. Y. Hussaini and R. G. Voigts, ed.), Springer Verlag, New York, 1988.

[23] J. T. Oden, L. Demkowitz, W. Rachowitz and T. A. Westerman, *Toward a universal h-p adaptive finite element strategy, Part 2. A posteriori error estimation*, Comp. Methods Appl. Mech. Engrg., 77 (1989), pp. 113–180.

[24] L. A. Oganesjan and L. A. Ruhovec, *Investigation of the convergence rate of variational–difference schemes for elliptic second order equations in a two dimensional domain with a smooth boundary*, Z.Vychisl. Mat. i Mat. Fiz., 9 (1969), pp. 1102–1120. (Translation in USSR Comput. Math. Math. Phy. 9 (1969), 153–188)

[25] W. Rachowitz, J. T. Oden and L. Demkowitz, *Toward a universal h-p adaptive finite element strategy, Part 3. Design of h-p meshes*, Comp. Methods Appl. Mech. Engrg., 77 (1989), pp. 181–212.

[26] B. Szabó, *Control of errors in finite element analysis: theoretical and practical aspects*, in FEM in the Design Process (J. Robinson, ed.), Robinson and associates, 1990, pp. 153–160.

[27] B. Szabó and I. Babuška, *Finite Element Analysis*, John Wiley & Sons, 1991.

[28] L. B. Wahlbin, *Local behaviour in finite element methods*, in Handbook of Numerical Analysis. Volume II (P. G. Ciarlet and J. L. Lions, ed.), North Holland, 1991, pp. 353–522.

[29] H. Zhong and P. Beckers, *Solution approximation error estimators for the finite element solution*, Report SA-140, Universite de Liege, 1990.

[30] J. Z. Zhu and O. C. Zienkiewicz, *Adaptive techniques in the finite element method*, Commun. Appl. Numer. Methods, 4 (1988), pp. 197–204.

[31] O. C. Zienkiewicz and J. Z. Zhu, *A simple error estimator and adaptive procedure for practical engineering analysis*, Internat. J. Numer. Meth. Eng., 24 (1987), pp. 337–357.

[32] O. C. Zienkiewicz and J. Z. Zhu, *The three R's of engineering analysis and error estimation and adaptivity*, in Reliability in Computational Mechanics (J. T. Oden, ed.), Elsevier Science Publisher B. V. (North Holland), 1990, pp. 95–114.

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